THE DISK AND SHELL METHOD

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In most calculus books there is little effort given to showing that the cylindrical shell method and disk method give the same value when computing the volume of a solid of revolution. Indeed it is not obvious that these two distinct methods should give the same result. In some texts this is demonstrated when the trapezoid bounded by the x-axis, $y = mx + b$, $x = a$ and $x = b$ is revolved about the y-axis.

In this paper we shall show that the cylindrical shell and disk methods give the same value if the region revolved about the y-axis is bounded by $y = f(x)$, $x = a$, $x = b$ and the x-axis, provided $f(x)$ is a differentiable function on $[a, b]$ and $f(x)$ is one-to-one. The proof is simple and uses two theorems which the students have recently learned (substitution formula and integration by parts). This proof can easily be included in a calculus course.

Consider the solid of revolution K produced by revolving the region bounded by $y = f(x)$, $x = a$, $x = b$ and the x-axis, about the y-axis. We use the shell method, which involves summing the volumes of cylindrical shells, to define the volume of K to be $\lim_{\|P\|\to 0}\sum_{i=1}^{n} 2\pi x_i f(x_i)\,\Delta x_i$. If $f(x)$ is differentiable on $[a, b]$ and hence continuous there, this limit exists and is equal to $\int_a^b 2\pi x f(x)\,dx$.

Suppose the region is bounded by the function $x = g(y)$, $y = c$, $y = d$ and the y-axis. In the disk method, which involves summing the volumes of disks, we consider

$$\lim_{\|P\|\to 0}\sum_{i=1}^{n}\pi[g(y_i)]^2\,\Delta y_i.$$

If $g(y)$ is continuous on $[c, d]$, this limit exists and is equal to $\int_c^d \pi[g(y)]^2\,dy$.

THEOREM. *Let $0 \leqslant a < b$, and let $y = f(x)$ be differentiable, nonnegative and $1-1$ on $[a, b]$, with $f(a) = c$ and $f(b) = d$ where $c < d$.* Also let $f'(x)$ be continuous on $[a, b]$ and let $x = g(y)$ iff $y = f(x)$. If R is the region bounded by $y = f(x)$, the x-axis, $x = a$ and $x = b$ and R is revolved about the y-axis, then the value obtained by using the disk method is equal to the value obtained by using the cylindrical shell method. Equivalently $\pi[b^2d - a^2c] - \int_c^d \pi[g(y)]^2\,dy = \int_a^b 2\pi x f(x)\,dx$.*

Proof. The region R can also be described as the region bounded by $x = m(y)$, $x = b$, $y = 0$ and $y = d$, where

$$m(y) = \begin{cases} a & \text{if } 0 \leqslant y < c, \\ g(y) & \text{if } c \leqslant y \leqslant d. \end{cases}$$

We observe from the way that $m(y)$ is defined that it is continuous on $[0, d]$. If we evaluate the volume obtained by revolving the region R about the y-axis by using the disk method, we find this to be $\int_0^d \pi b^2\,dy - \int_0^d \pi m(y)\,dy$. This is equal to

$$\int_0^d \pi b^2\,dy - \int_0^c \pi a^2\,dy - \int_c^d \pi[g(y)]^2\,dy = \pi[b^2d - a^2c] - \int_c^d \pi[g(y)]^2\,dy.$$

By the substitution formula, the latter expression is equal to $\pi[b^2d - a^2c] - \int_a^b \pi x^2 f'(x)\,dx$. A straightforward application of the integration by parts formula and algebraic simplification shows that

$$\int_a^b 2\pi x f(x)\,dx = \pi[b^2d - a^2c] - \int \pi x^2 f'(x)\,dx,$$

and the argument is complete.

*The result is true in case $d < c$, but a slight alteration is needed in the argument.