A study of $e^{ix}, e^{i\pi} = -1$

-- Visualizing $e^{i\pi} = -1$ by reexamining the definition of e^{ix} --

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1. Introduction (aim of the study)

In modern mathematics, e^{ix} is defined through complex function theory. $e^{i\pi} = -1$, known as "Euler's equation," holds as a property of e^{ix}. This is a very wonderful relation for the important mathematical constants $e, i, \pi, -1$ which appear in the equation, but the meaning of the equation is unclear. e^{ix} was introduced by Euler, so this study examines the meaning of the equation by examining his thinking.

2. Details of the study

2.1 Investigation of definitions and approach in complex function theory

2.1.1 Definition of e^{ix} and explanation of $e^{i\pi} = -1$ in modern complex function theory

The reasoning in complex function theory is as follows:

・Definition of arithmetic operations using complex numbers

・Definition of functions using power series

As an example of a function, $e^{z} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots, z \in C$

・If $z = ix, x \in R$, $e^{ix} = \cos x + i \sin x$

・In particular, if $x = \pi$, then $e^{i\pi} = -1$.

2.1.2 Approach to $e^{i\pi} = -1$ in complex function theory

Using the method above, $e^{i\pi} = -1$ is logically derived through the transformation of equations.

2.2 Investigation of Euler's definitions and approach

2.2.1 Overview of definition of e^{ix} by Euler

Since Euler's proof method differs from modern proof methods, the overview will be explained based on Leonhard Euler (1748, 1, pp. 85–107). The notation used in Euler's writings is different from that used today. Misleading notations have been replaced with the modern notation.

	References	Modern notation
Imaginary number symbol	$\surd{-}\mathrm{I}$	i
Large natural number	i	N
Factorial	$1{\cdot}2{\cdot}3{\cdot}4\cdots$	$5!$
Trigonometric function	$\int in.$ 、 $co\int.$	sin 、 cos
	$\left(\int in.v\right)^n$ 、 $\left(co\int.v\right)^n$	$\sin^n v$ 、 $\cos^n v$
Limit symbols	None	lim 、 $N \to \infty$

Table1 Notation correspondence table

Introduction of Euler's constant e

Let a be a constant, and let ω be a sufficiently small positive number. a^{ω} becomes a number slightly larger than 1, and there exists k satisfying $a^{\omega}=1+k\omega$. Let N be a sufficiently large natural number.

$$a^{\omega N}=(1+k\omega)^{N}=1+\frac{N}{1}k\omega+\frac{N(N-1)}{2!}k^{2}\omega^{2}+\cdots$$

Set $\omega N=x$. x becomes a positive real number.

$$a^{x}=\left(1+\frac{kx}{N}\right)^{N}=1+\frac{1}{1}kx+\frac{1\cdot(N-1)}{2!N}k^{2}x^{2}+\cdots$$

$\lim_{N\to\infty}\frac{N-1}{N}=1, \lim_{N\to\infty}\frac{N-1}{2N}=\frac{1}{2}, \lim_{N\to\infty}\frac{N-2}{3N}=\frac{1}{3},\cdots$ (Note: The symbol $\lim_{N\to\infty}$ is not used in the reference.)

This becomes: $a^{x}=1+\frac{kx}{1}+\frac{k^{2}x^{2}}{2!}+\frac{k^{3}x^{3}}{3!}+\frac{k^{4}x^{4}}{4!}+\cdots$

Letting $x=1, k=1$, then $a=1+\frac{1}{1}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots=2.71828\cdots$

Here, use e to indicate a. This is the definition $e=\lim_{N\to\infty}\left(1+\frac{1}{N}\right)^{N}$.

Introduction of trigonometric functions

When the radius of a circle is set to 1, half the circumference is 3.14159… This is indicated as π. (Note: This leads to the introduction of radian measure, but the reason for its introduction is not described.) There are indications such as $\sin 0\pi=0, \sin\frac{\pi}{2}=1, \sin\frac{3\pi}{2}=-1,\ldots$ (Note: Due to $\sin\frac{3\pi}{2}=-1$, it follows that Euler recognized negative numbers.) Furthermore, various formulas for trigonometric functions are also indicated.

$$\cos z=\sin\left(\frac{\pi}{2}-z\right), \sin z=\sin\left(\frac{\pi}{2}-z\right), \sin^{2}z+\cos^{2}z=1, \tan z=\frac{\sin z}{\cos z}$$

$$\sin(y+z)=\sin y\cos z+\cos y\sin z$$

Aside from these, there are double angle formulas for trigonometric functions, sum and product formulas, and others.

Introduction of Euler's formula

Since $\sin^{2}x+\cos^{2}x=1$, $(\cos x+i\sin x)(\cos x-i\sin x)=1$

More generally, letting n be a natural number, this becomes $(\cos x\pm i\sin x)^{n}=\cos nx\pm i\sin nx$. It can be assumed that $\sin x=x$ and $\cos x=1$ for a sufficiently small positive real number x. (Note: When written in modern style, this is $\lim_{x\to+0}\frac{\sin x}{x}=1$, but no proof is provided.)

Let N be a sufficiently large natural number.

$$(\cos x\pm i\sin x)^{N}=\cos Nx\pm i\sin Nx$$

$$\cos Nx=\left\{(\cos x+i\sin x)^{N}+(\cos x-i\sin x)^{N}\right\}\Big/2$$

$$\sin Nx=\left\{(\cos x+i\sin x)^{N}-(\cos x-i\sin x)^{N}\right\}\Big/2i$$

Set $x=\frac{v}{N}$. v becomes a real number.

$$\cos v=\left\{\left(\cos\frac{v}{N}+i\sin\frac{v}{N}\right)^{N}+\left(\cos\frac{v}{N}-i\sin\frac{v}{N}\right)^{N}\right\}\Big/2$$

$$\sin v=\left\{\left(\cos\frac{v}{N}+i\sin\frac{v}{N}\right)^{N}-\left(\cos\frac{v}{N}-i\sin\frac{v}{N}\right)^{N}\right\}\Big/2i$$

$\frac{v}{N}$ is a sufficiently small positive real number, so it can be assumed that $\sin\frac{v}{N}=\frac{v}{N}$ and $\cos\frac{v}{N}=1$.

$\cos v=\lim_{N\to\infty}\left\{\left(1+\frac{iv}{N}\right)^N+\left(1-\frac{iv}{N}\right)^N\right\}\Big/2$, $\sin v=\lim_{N\to\infty}\left\{\left(1+\frac{iv}{N}\right)^N-\left(1-\frac{iv}{N}\right)^N\right\}\Big/2i$

$e^v=\lim_{N\to\infty}\left(1+\frac{v}{N}\right)^N$, so replacing v with iv yields $\cos v=\frac{e^{iv}+e^{-iv}}{2}$ and $\sin v=\frac{e^{iv}-e^{-iv}}{2i}$. Therefore, the equation becomes $e^{iv}=\cos v+i\sin v$.

Investigation of the notation $e^{i\pi}=-1$

The author investigated works by Euler, from Leonhard Euler (1748, 1) to Leonhard Euler (1770). The references are also available within the Euler Archive https://scholarlycommons.pacific.edu/euler/, so other references at that site were also investigated. However, $e^{i\pi}=-1$ was not found.

2.2.2 Euler's approach to e^{iv} and $e^{i\pi}=-1$

Since Euler replaced v in $e^v=\lim_{N\to\infty}\left(1+\frac{v}{N}\right)^N$ with iv, this is a transformation of an equation under the assumption that function variables can be replaced from real numbers to complex numbers. This is a logical equation transformation. Since the equation $e^{i\pi}=-1$ was not found in the reference, it is not clear who developed the equation $e^{i\pi}=-1$.

2.3 Problem with $e^{i\pi}=-1$

$e^{i\pi}=-1$ is a very wonderful equation, but it is unclear what it means.

2.4 Proposed definition of e^{ix} and visualization of $e^{i\pi}=-1$

The following function is defined in the complex plane.

$f(x)=\{$Coordinates when argument is x on the unit circle in the complex plane$\}$.

We can set $f(x)=\cos x+i\sin x$. The properties of $f(x)$ are:

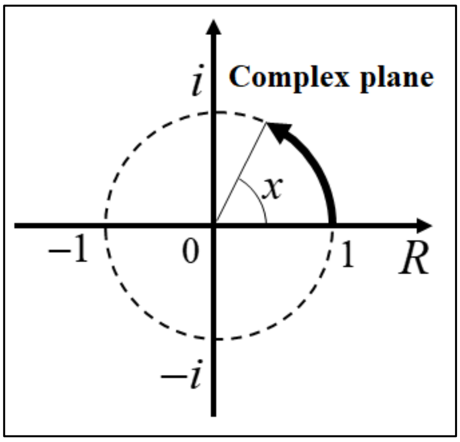

Fig. 1: Unit circle in complex plane

① $f(x_1+x_2)=\cos(x_1+x_2)+i\sin(x_1+x_2)$

$$=(\cos x_1+i\sin x_1)(\cos x_2+i\sin x_2)$$

$$=f(x_1)f(x_2)$$

② $\frac{d}{dx}f(x)=\frac{d}{dx}(\cos x+i\sin x)$

$$=i(\cos x+i\sin x)$$

$$=i\,f(x)$$

③ $f(x)=\cos x+i\sin x$

$$=\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\cdots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)$$

$$= 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \cdots$$

It is evident, in light of properties ①, ②, and ③, that $f(x)$ has the same properties as the exponential function e^{ax} taking a to be a real number constant. Thus the notation $f(x) = e^{ix}$ is used, taking the real exponential function for reference. This means that "the locus of the circumference of the unit circle on the complex plane and the locus of the real exponential function have the same properties."

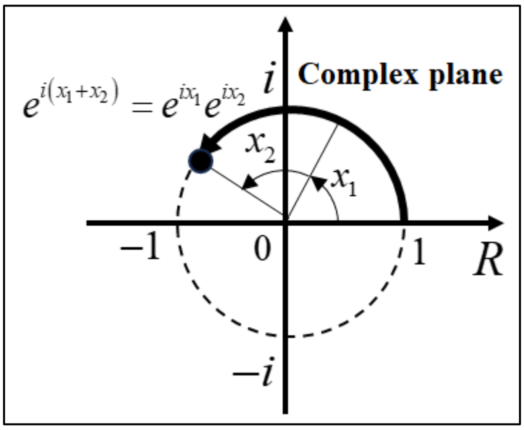

Fig. 2: Unit circle in complex plane

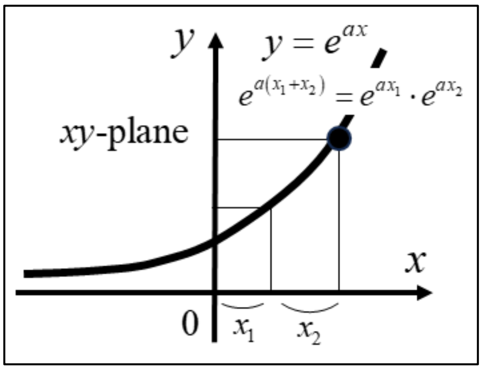

Fig. 3: Exponential function

Here, if we set $x = \pi$, then $e^{i\pi} = -1$. Putting this into words, we have: "On the unit circle in the complex plane, the coordinate when the argument is π is -1." With this, visualization of $e^{i\pi} = -1$ has been achieved.

The approach to each symbol can be interpreted as follows:

i: Imaginary number, $i^2 = -1$

π : Argument, $\pi = 3.14\cdots$

e : Function symbol, $e \neq 2.71\cdots$

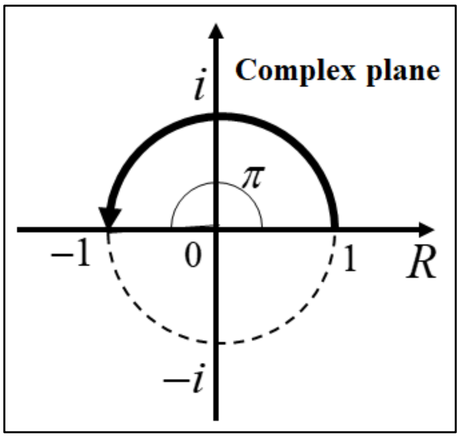

Fig. 4: $e^{i\pi} = -1$

3. Conclusion

By considering a definition specialized for e^{ix}, it was possible to understand the meaning of $e^{i\pi} = -1$. However, a definition specialized for complex functions such as $\sin(ix)$ and $\cos(ix)$ was also considered, but no special properties were found.

References

[1]Leonhard Euler, Introductio in analysin infinitorum volume 1, https://scholarlycommons.pacific.edu/euler-works/101/, 1748, accessed on 2023/11/30

[2]Leonhard Euler,Introductio in analysin infinitorum volume 2, https://scholarlycommons.pacific.edu/euler-works/102/, 1748, accessed on 2023/11/30

[3]Leonhard Euler,Institutiones calculi differentialis cum eius usu in analysi fifinitorum, https://scholarlycommons.pacific.edu/euler-works/212/, 1755, accessed on 2023/11/30

[4]Leonhard Euler, Institutionum calculi integralis volume primum, https://scholarlycommons.pacific.edu/euler-works/342/, 1768, accessed on 2023/11/30

[5]Leonhard Euler, Institutionum calculi integralis volume secundum,

https://scholarlycommons.pacific.edu/euler-works/366/,
1769, accessed on 2023/11/30
[6]Leonhard Euler, Institutionum calculi integralis volume tertium,
https://scholarlycommons.pacific.edu/euler-works/385/,
1770, accessed on 2023/11/30