A Study of Definition of the Definite Integral

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1. Objective of the study

According to Richard Courant(1989,P.123-125), the definition of the definite integral using piecewise quadrature is:

"$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(\xi_i\right)\Delta x_i = \int_a^b f\left(x\right)dx$

Where $x_{i-1} < \xi_i < x_i, \Delta x_i = x_i - x_{i-1}, \max\left|\Delta x_i\right| \to 0\left(n\to\infty\right)$".

Incidentally, if we look closely at the definition of the definite integral, it is evident that there is a gap between the representation methods on the left and right side. The "=" sign can be interpreted as indicating definition, but if it is interpreted in the sense of "=" in an equation, then the rule corresponding to the operation rule where "left side = right side" cannot be explained. Thus, to find a rule corresponding to the operation rule between the two sides, the author decided to survey the work of Leibniz, and investigate the origin of the integration symbol. In this study, the work of Leibniz is explained, and at the same time, an approach to definition of the definite integral suggested by that is proposed.

Furthermore, students learn various integrals in university. Among those, Richard Courant(1989, P.388) presents the following double integral.

"*First we take the region* R *as a rectangle* $a \le x \le b$, $\alpha \le y \le \beta$ *in the* x, y *-plane and consider a continuous function* $f\left(x,y\right)$ *in* R. ………..

$\iint_R f\left(x,y\right)dR = \int_\alpha^\beta dy\int_a^b f\left(x,y\right)dx$".

On the other hand, $\iint_R f\left(x,y\right)dR = \int_\alpha^\beta\int_a^b f\left(x,y\right)dxdy$ holds. if we set $\int_a^b f\left(x,y\right)dx = F\left(y\right)$, then the equation $\int_\alpha^\beta F\left(y\right)dy = \int_\alpha^\beta dyF\left(y\right)$ holds. In this equation, the commutative law holds between $F\left(y\right)$ and dy . Therefore, grounds for why the commutative law holds are also proposed.

2. Content of the study

2.1. Shift in usage of d and $\int$ by Leibniz

October 1675,No.1

For the problem of quadrature, he uses the symbol "omn." devised by Cavalieri.

Gottfried Wilhelm Leibniz (2008,P.263) presents the following formula

" $\text{omn.}\overline{yx}\,\text{ad}\,x \sqcap \frac{b^2c}{2} - \text{omn.}\overline{\frac{x^2}{2}}\,\text{ad}\,y$ ".

"omn." is an abbreviation for the Latin "omnes" and means "all lines" in the method of indivisibles. " $\overline{xy}$ " means " $\left(xy\right)$ ". " $ad\ x$ " means "related to x ". " $\sqcap$ " means "=".

October 1675,No.2

Gottfried Wilhelm Leibniz (2008,P.292) presents the following contents "*Utile erit scribe* $\int.$ *pro omn. ut* $\int l$ *pro omn.* l" . This means "*it should be convenient to write* $\int$ *instead of omn.*" in English. The symbol $\int$ was introduced for the first time in the history of mathematics.

July 1676

Gottfried Wilhelm Leibniz (2008,PP.599-600) presents the following formulas "$\frac{t}{y} \sqcap \frac{d\bar{x}}{d\bar{y}}$. *Ergo* $\frac{d\bar{x}}{d\bar{y}} \sqcap \frac{n}{y-x}$. $d\bar{x}y - xd\bar{x} \sqcap d\bar{y}n$. *Ergo* $\int d\bar{x}y - \int \overline{xd\bar{x}} \sqcap n\int d\bar{y}$".

$\frac{t}{y} \sqcap \frac{d\bar{x}}{d\bar{y}}$: This is the first expression in the history of mathematics that differential symbol was placed in the denominator.

$\int d\bar{x}y, \int \overline{xd\bar{x}}$:This is the first expression in the history of mathematics that combines $\int$ and d .

2.2. Discussion of the shift in usage of d and $\int$

The method of changing $\frac{d\bar{x}}{d\bar{y}} \sqcap \frac{n}{y-x}$ to $d\bar{x}y - xd\bar{x} \sqcap d\bar{y}n$ is to multiply both sides of $\frac{d\bar{x}}{d\bar{y}} \sqcap \frac{n}{y-x}$ by $d\bar{y}(y-x)$. Therefore, it can be interpreted that $d\bar{x} \times y - x \times d\bar{x} \sqcap d\bar{y} \times n$.

The equality of the sum of difference expression $\left(d\bar{x} \times y - x \times d\bar{x}\right)$ with the sum of difference expression $\left(d\bar{y} \times n\right)$ is indicated as $\int d\bar{x}y - \int \overline{xd\bar{x}} \sqcap n\int d\bar{y}$. In $\int d\bar{x}y$ for example, It is appropriate to interpret the summation symbol $\int$ as not acting only on $d\bar{x}$ or only on y , but as acting on $d\bar{x} \times y$ .This can be interpreted as $\int\left(d\bar{x} \times y\right) - \int\left(x \times d\bar{x}\right) \sqcap n \times \int d\bar{y}$.

Leibniz struggled to realize a symbolism with d and $\int$, but "$\times$" is not written inside the integral symbol. It is likely that "$\times$" was omitted. This raises the question of why Leibniz omitted "$\times$". According to René Descartes(1637,P.3), the four rules of arithmetic are expressed as $a+b$, $a-b$, ab , $\frac{a}{b}$, and "$\times$" are omitted for multiplication. That is, it is likely that Leibniz adopted the notational methods of Descartes.

3. Research results

For these reasons, it is proposed to revise the definition of the definite integral as follows.

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(\xi_i\right)\Delta x_i = \int_a^b \left(f\left(x\right) \times dx\right) = \int_a^b f\left(x\right)dx$$

The meaning of $\int_a^b \left(f\left(x\right) \times dx\right)$ is to find the area by multiplying the infinitesimal difference dx by the value $f\left(x\right)$ at x on an infinitesimal interval, and this is enclosed in parentheses. Also, the source of the integral is the meaning of a sum (Latin "summa"), and thus it means "find the sum from

a to b ". Subsequently, omitting $\times$ and $(\ )$ yields the conventional expression.

4. Properties and greater strictness of " $\times$ "

Consider the properties of " $\times$ " in $\int_a^b \left(f(x) \times dx \right)$.

Integrating by substitution, this can be written as follows:

$$\int_a^b \left(f(x) \times dx \right) = \int_\alpha^\beta \left(f\left(g(t)\right) \times \frac{dx}{dt} \times dt \right) = \int_\alpha^\beta f\left(g(t)\right) \frac{dx}{dt}\, dt$$

Therefore, " $\times$ " has the same properties as multiplication. Due to the commutative law of multiplication, the equation $\int_a^b \left(f(x) \times dx \right) = \int_a^b \left(dx \times f(x) \right)$ holds.

Now, in the previous section, dx was explained as an infinitesimal difference, but the question arises: "how great is the difference of an infinitesimal difference?" If, in contrast, a decimal description is used, then the number of zeroes after the decimal point is infinite, and it's impossible to specify a precise value of the infinitesimal difference. Therefore, an infinitesimal difference is an abstract concept. Operations are done on an abstract infinitesimal difference, so " $\times$ " is multiplication of abstract concepts.

5. Learning effect due to understanding the definition

Now let us explain the learning effect due to incorporating the definition of the definite integral explained in this paper into the learning process of students.

When students learn the definition of the definite integral based on piecewise quadrature, incorporating the material explained in this paper is anticipated to enable understanding, without any stress, of the sophisticated integrals that will be learned if the student proceeds on to university in a scientific or technical field. They should be able to understand the meaning of symbols for double integrals, surface integrals, line integrals, complex integrals, integrals on vectors, and (in mathematics) Lebesgue integrals. Also, with previous instructional methods, there is no need to adequately explain the definition of the definite integral, and thus there is a possibility that some students do not correctly understand the definition part. For this sort of high school student, it is expected that using the method explained in this study will enable them to correctly understand the definition of the definite integral.

References

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