Properties of the Laurent expansion

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Theorem 1: Condition where term degrees skip by *n* in a Laurent expansion

Let $w = f(z)$ be a function on the complex plane, and assume it is regular on the domain D $:0 \leqq R_1 < |z-\alpha| < R_2 \leqq \infty$. Assume that $w = f(z)$ is Laurent expandable, and let $f(z) = \sum_{\nu=-\infty}^{\infty} c_\nu (z-\alpha)^\nu$. If n is a natural number equal to or greater than 2, a necessary and sufficient condition for term degrees to skip by n, i.e., $f(z) = \sum_{\nu=-\infty}^{\infty} c_{n\nu} (z-\alpha)^{n\nu}$, is that when we convert to polar form by using $z = \alpha + re^{i\theta} (0 \leqq \theta \leqq 2\pi)$ and set $f(z) = F(r,\theta)$, then $F(r,\theta)$ becomes identically $F(r,\theta) = F(r,\theta + 2\pi/n)$.

Theorem 2: Number of points causing periodicity in the function

Let $w = f(z)$ be an entire function defined on the complex plane. Convert $f(z)$ to polar form by using $z = \alpha + re^{i\theta}$ at point α, and let $f(z) = F_\alpha(r,\theta)$. Let n be a natural number greater than or equal to 2. Then in the complex plane, one of the following holds regarding the number of α where $F_\alpha(r,\theta)$ identically satisfies $F_\alpha(r,\theta) = F_\alpha(r,\theta + 2\pi/n)$:

① None exist
② Only 1 exists
③ An infinite (countable) number exist at equal intervals on a straight line, and at each point $F_\alpha(r,\theta) = F_\alpha(r,\theta + \pi)$

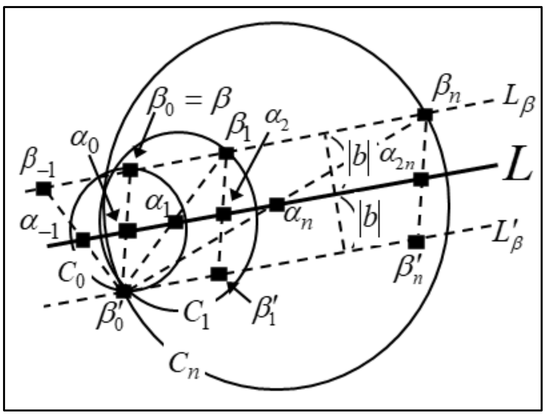
holds.

Theorem 3: If there are an infinite number of points causing periodicity

Let $w = f(z)$ be an entire function defined on the complex plane. Convert $f(z)$ to polar form by using $z = \alpha + re^{i\theta}$ at point α, and let $f(z) = F_\alpha(r,\theta)$. Assume there are an infinite (countable) number of points α such that $F_\alpha(r,\theta) = F_\alpha(r,\theta + \pi)$ at equal intervals on the straight line L. Designate this sequence of points as $\{\alpha_n\}(-\infty \leq n \leq \infty)$.

3.1 Periodicity of function

Let $w_\beta = f(\beta)$ at an arbitrary point β on the complex plane. Rearrange the order of the numbers n in $\{\alpha_n\}$ and let the point nearest to β in the point sequence $\{\alpha_n\}$ be α_0. For points z other than point β satisfying $w_\beta = f(z)$, $w_\beta = f(\beta_n)$ in the point sequence $\{\beta_n\}$ $(\beta_0 = \beta)$ with twice the interval of the interval in the point sequence $\{\alpha_n\}$, on the straight line L_β which passes through point β parallel to line L.



Also, set $\beta_n = \alpha_{2n} + re^{i\theta}$. Using $\beta'_n = \alpha_{2n} + re^{i(\theta+\pi)}$, it is possible to create the point sequence $\{\beta'_n\}$, and these points satisfy $w_\beta = f(\beta'_n)$.

The point sequence $\{\beta'_n\}$ is located on the straight line L'_β on the opposite side, with an equal distance between the line L and the line L_β, and an infinite (countable) number of points exist at the same interval as the interval for $\{\beta_n\}$.

The point β is an arbitrary point, so if we set $\alpha_1 - \alpha_0 = a$, then $f(z) = f(z+2a)$ holds identically.

3.2 Periodicity of domain

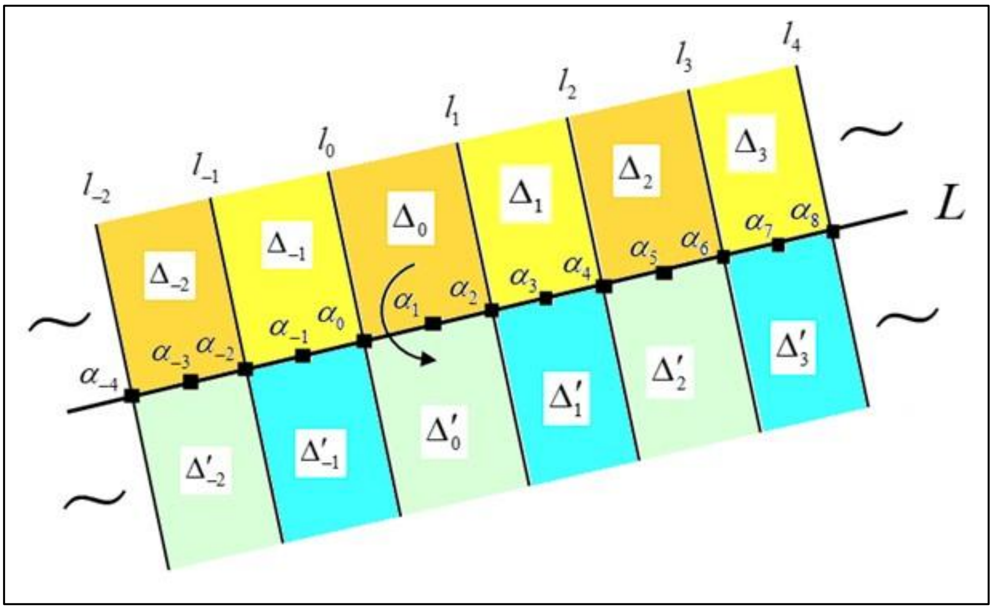
Let $\cdot\cdot, l_{-1}, l_0, l_1, l_2 \cdot\cdot$ be lines perpendicular to L, passing through the point sequence on L, $\cdot\cdot, \alpha_{-2}, \alpha_0, \alpha_2, \alpha_4, \cdots$. Let Δ_0 be the domain on one side of L surrounded by l_0, l_1. A sequence of domains $\{\Delta_n\}(-\infty \le n \le \infty)$ is defined using the same method. Let Δ'_0 be the domain on the other side of L surrounded by l_0, l_1. A sequence of domains $\{\Delta'_n\}(-\infty \le n \le \infty)$ is defined using the same method.

Due to "3.1 Periodicity of function", the following hold:

$$\cdot\cdot = f(\Delta_{-2}) = f(\Delta_{-1}) = f(\Delta_0) = f(\Delta_1) = f(\Delta_2) = \cdot\cdot$$

$$\cdot\cdot = f(\Delta'_{-2}) = f(\Delta'_{-1}) = f(\Delta'_0) = f(\Delta'_1) = f(\Delta'_2) = \cdot\cdot$$

The domain obtained by rotating Δ_0 by 180° about α_1 is Δ'_0. Because $F_{\alpha 1}(r,\theta) = F_{\alpha 1}(r,\theta+\pi)$, $f(\Delta_0) = f(\Delta'_0)$ holds. Similarly, $f(\Delta_n) = f(\Delta'_n), (-\infty \le n \le \infty)$ holds.



3.3 An alternative interpretation of periodicity

In 3.1 and 3.2, periodicity was expressed using even numbers, e.g., α_{-2} and α_0, α_0 and α_2, α_2 and α_4, but periodicity can also be expressed with odd numbers, e.g., α_{-1} and α_1, α_1 and α_3, α_3 and α_5.

Prediction: Regarding the non-trivial zero points of the ς function

$\varsigma(s)(s-1)$ becomes a regular function over the entire complex plane. Assume that we have further converted the function, and made it into a function that satisfies $F_\alpha(r,\theta) = F_\alpha(r,\theta+\pi)$. If a single point can be found corresponding to a non-trivial zero point in Δ_0 or on L, then an infinite number of similar points exist due to periodicity, and thus this should yield a proof of the Riemann hypothesis.

The n th zero point is evident from $F_\alpha(r,\theta) = F_\alpha(r,\theta+\pi)$, so if an inverse conversion is done, the n th non-trivial zero point of $\varsigma(s)$ will likely be evident.