A study of 4-dimensional complex space

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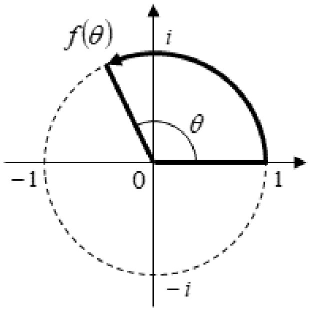
1. Introduction

1.1 Introduction (conclusion)

Research was conducted on 4-dimensional complex space and beautiful formulas were obtained, but examples of practical applications have not been found.

1.2 Introduction (interpretation of complex numbers)

In “A study of $e^{ix}, e^{i\pi} = -1$”, it was explained that $e^{i\pi} = -1$ can be translated into words as follows: “on the unit circle in the complex plane, the coordinate when the argument is π is -1.” By multiplying by the complex number i, the values change as follows: $1 \to i \to -1 \to -i \to 1$. This transformation corresponds to the action of adding an argument of $\frac{\pi}{2}$, so it can also be interpreted as a rotation group. Extending this, we consider the rotation group generated by the complex number i and the rotation group generated by the complex number j. The composition of i-rotation and j-rotation is called ij-rotation. Changing the order of composition, we let “i-rotation and j-rotation = j-rotation and i-rotation”. Based on this assumption, we construct a space and operations.



2. Content of study

(Definition of complex space)

Let Z_i, Z_j each be represented in complex form as $Z_i = \alpha_i + \beta_i i, Z_j = \alpha_j + \beta_j j \quad (\alpha_i, \beta_i, \alpha_j, \beta_j \in R)$.

Assume $i^2 = -1, j^2 = -1$. $(ij)^2 = 1$

$$Z_i Z_j = (\alpha_i + \beta_i i)(\alpha_j + \beta_j j) = \alpha_i\alpha_j + \alpha_j\beta_i i + \alpha_i\beta_j j + \beta_i\beta_j ij$$

Write $Z_i Z_j = Z = a + bi + cj + dij\,(a,b,c,d \in R)$.

Due to the assumption that “i-rotation and j-rotation = j-rotation and i-rotation,” the equation $ij = ji$ holds.

When the representation $Z = a + bi + cj + dij\,(a,b,c,d \in R)$ is adopted, the result is 4-dimensional complex space.

(Definition of operations)

For $Z_1 = a_1 + b_1 i + c_1 j + d_1 ij$ and $Z_2 = a_2 + b_2 i + c_2 j + d_2 ij$, $(a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in R)$,

(Addition, subtraction, and multiplication)

$$Z_1 \pm Z_2 = (a_1 \pm a_2) + (b_1 \pm b_2)i + (c_1 \pm c_2)j + (d_1 \pm d_2)ij$$

$$\begin{aligned} Z_1 \times Z_2 = &(a_1a_2 - b_1b_2 - c_1c_2 + d_1d_2) + (a_1b_2 + b_1a_2 - c_1d_2 - d_1c_2)i \\ &+ (a_1c_2 - b_1d_2 + c_1a_2 - d_1b_2)j + (a_1d_2 + b_1c_2 + c_1b_2 + d_1a_2)ij \end{aligned}$$

The above define addition, subtraction, and multiplication. Note that $Z_1 \times Z_2 = 0$ when any of the following conditions are satisfied:

Condition 1 $(a_1 = -d_1, b_1 = c_1)$, Condition 2 $(a_1 = d_1, b_1 = -c_1)$,

Condition 3 $\left(a_2=-d_2, b_2=c_2\right)$, Condition 4 $\left(a_2=d_2, b_2=-c_2\right)$.
(Zero factors)
Any number Z satisfying $a=-d, b=c$ or $a=d, b=-c$ has the same properties as 0.
We use the notation $\mathbf{O}_Z=\{Z: Z=(a, b, c, d), a=-d, b=c \text{ or } a=d, b=-c\}$.
(Division)

$$\frac{Z_2}{Z_1}=\frac{a_2+b_2 i+c_2 j+d_2 ij}{a_1+b_1 i+c_1 j+d_1 ij}$$

Division is defined as above. However, division is not valid when $Z_1 \in \mathbf{O}_Z$.

(Three basic laws for operations)

The commutative law, associative law, and distributive law all hold.

(Representing complex numbers as products)

If $Z \neq 0$, with $\alpha_i, \beta_i, \alpha_j, \beta_j, \alpha_{ij}, \beta_{ij} \in R$, and $\alpha_j \geq 0, \alpha_{ij} \geq 0,\left|\alpha_{ij}\right| \geq\left|\beta_{ij}\right|$,
it can be represented as a product of i, j, ij:

$$Z=a+bi+cj+dij=\left(\alpha_i+\beta_i i\right)\left(\alpha_j+\beta_j j\right)\left(\alpha_{ij}+\beta_{ij} ij\right)$$

Furthermore, the following relationships are satisfied:

$$a^2+b^2+c^2+d^2=\left(\alpha_i^2+\beta_i^2\right)\left(\alpha_j^2+\beta_j^2\right)\left(\alpha_{ij}^2+\beta_{ij}^2\right)$$

$$a^2+b^2-c^2-d^2=\left(\alpha_i^2+\beta_i^2\right)\left(\alpha_j^2-\beta_j^2\right)\left(\alpha_{ij}^2-\beta_{ij}^2\right)$$

$$a^2-b^2+c^2-d^2=\left(\alpha_i^2-\beta_i^2\right)\left(\alpha_j^2+\beta_j^2\right)\left(\alpha_{ij}^2-\beta_{ij}^2\right)$$

(Absolute value)

For $Z=a+bi+cj+dij$, the absolute values $|Z|,\|Z\|$ are defined as:

$$|Z|=\sqrt{a^2+b^2+c^2+d^2}, \quad \|Z\|=\sqrt[4]{(a-d)^2+(b+c)^2} \sqrt[4]{(a+d)^2+(b-c)^2}$$

If $Z_1=a_1+b_1 i+c_1 j+d_1 ij$ and $Z_2=a_2+b_2 i+c_2 j+d_2 ij$, then $\left\|Z_1 Z_2\right\|=\left\|Z_1\right\|\left\|Z_2\right\|$ holds,
but $\left|Z_1 Z_2\right|=\left|Z_1\right|\left|Z_2\right|$ does not generally hold.
Alternate expression for indicating absolute value $\|Z\|$:

$$\begin{aligned}\|Z\| &=\sqrt[4]{(a+bi+cj+dij)(a+bi-cj-dij)(a-bi+cj-dij)(a-bi-cj+dij)} \\ &=\sqrt{\alpha_i^2+\beta_i^2} \sqrt{\alpha_j^2+\beta_j^2} \sqrt{\alpha_{ij}^2-\beta_{ij}^2}\end{aligned}$$

(Polar coordinate representation)

This becomes:

$$Z=\|Z\| \frac{\alpha_i+\beta_i i}{\sqrt{\alpha_i^2+\beta_i^2}} \frac{\alpha_j+\beta_j j}{\sqrt{\alpha_j^2+\beta_j^2}} \frac{\alpha_{ij}+\beta_{ij} ij}{\sqrt{\alpha_{ij}^2-\beta_{ij}^2}}$$

Therefore, if $a=d, b=-c$ or $a=-d, b=c$ does not hold, i.e., if Z is not a zero element, then there exist $\theta_i, \theta_j, \theta_{ij}$ satisfying the following conditions:

$$Z=\|Z\|\left(\cos \theta_i+i \sin \theta_i\right)\left(\cos \theta_j+j \sin \theta_j\right)\left(\cosh \theta_{ij}+ij \sinh \theta_{ij}\right)$$

$$0 \leq \theta_i \leq 2\pi, -\frac{\pi}{2} \leq \theta_j \leq \frac{\pi}{2}, -\infty<\theta_{ij}<\infty$$

The expression $Z=\|Z\|e^{i\theta_i+j\theta_j+ij\theta_{ij}}$ is also possible. Furthermore, $\theta_i,\theta_j,\theta_{ij}$ can be determined uniquely.

(Finding $\alpha_i,\beta_i,\alpha_j,\beta_j,\alpha_{ij},\beta_{ij}$ and $\theta_i,\theta_j,\theta_{ij}$ from a,b,c,d)

For $Z=a+bi+cj+dij=(\alpha_i+\beta_i i)(\alpha_j+\beta_j j)(\alpha_{ij}+\beta_{ij}ij)$,

$$\cos\theta_i=\frac{\alpha_i}{\|Z\|}\sqrt{\left(\alpha_j{}^2+\beta_j{}^2\right)\left(\alpha_{ij}{}^2-\beta_{ij}{}^2\right)}=\pm\frac{1}{\sqrt{2}\|Z\|}\sqrt{\|Z\|^2+a^2-b^2+c^2-d^2}$$

$$\sin\theta_i=\frac{\beta_i}{\|Z\|}\sqrt{\left(\alpha_j{}^2+\beta_j{}^2\right)\left(\alpha_{ij}{}^2-\beta_{ij}{}^2\right)}=\pm\frac{1}{\sqrt{2}\|Z\|}\sqrt{\|Z\|^2-a^2+b^2-c^2+d^2}$$

$$\cos\theta_j=\frac{\alpha_j}{\|Z\|}\sqrt{\left(\alpha_i{}^2+\beta_i{}^2\right)\left(\alpha_{ij}{}^2-\beta_{ij}{}^2\right)}=\frac{1}{\sqrt{2}\|Z\|}\sqrt{\|Z\|^2+a^2+b^2-c^2-d^2}$$

$$\sin\theta_j=\frac{\beta_j}{\|Z\|}\sqrt{\left(\alpha_i{}^2+\beta_i{}^2\right)\left(\alpha_{ij}{}^2-\beta_{ij}{}^2\right)}=\pm\frac{1}{\sqrt{2}\|Z\|}\sqrt{\|Z\|^2-a^2-b^2+c^2+d^2}$$

$$\cosh\theta_{ij}=\frac{\alpha_{ij}}{\|Z\|}\sqrt{\left(\alpha_i{}^2+\beta_i{}^2\right)\left(\alpha_j{}^2+\beta_j{}^2\right)}=\frac{1}{\sqrt{2}\|Z\|}\sqrt{a^2+b^2+c^2+d^2+\|Z\|^2}$$

$$\sinh\theta_{ij}=\frac{\beta_{ij}}{\|Z\|}\sqrt{\left(\alpha_i{}^2+\beta_i{}^2\right)\left(\alpha_j{}^2+\beta_j{}^2\right)}=\pm\frac{1}{\sqrt{2}\|Z\|}\sqrt{a^2+b^2+c^2+d^2-\|Z\|^2}$$

If $\alpha_j{}^2+\beta_j{}^2=1,\alpha_{ij}{}^2-\beta_{ij}{}^2=1$, then

$$\alpha_i=\pm\frac{1}{\sqrt{2}}\sqrt{\|Z\|^2+a^2-b^2+c^2-d^2}\ ,\beta_i=\pm\frac{1}{\sqrt{2}}\sqrt{\|Z\|^2-a^2+b^2-c^2+d^2}$$

If $\alpha_i{}^2+\beta_i{}^2=1,\alpha_{ij}{}^2-\beta_{ij}{}^2=1$, then

$$\alpha_j=\frac{1}{\sqrt{2}}\sqrt{\|Z\|^2+a^2+b^2-c^2-d^2}\ ,\beta_j=\pm\frac{1}{\sqrt{2}}\sqrt{\|Z\|^2-a^2-b^2+c^2+d^2}$$

If $\alpha_i{}^2+\beta_i{}^2=1,\alpha_j{}^2+\beta_j{}^2=1$, then

$$\alpha_{ij}=\frac{1}{\sqrt{2}}\sqrt{a^2+b^2+c^2+d^2+\|Z\|^2}\ ,\beta_{ij}=\pm\frac{1}{\sqrt{2}}\sqrt{a^2+b^2+c^2+d^2-\|Z\|^2}$$

$$\theta_{ij}=\frac{1}{2}\log\frac{\alpha_{ij}+\beta_{ij}}{\alpha_{ij}-\beta_{ij}}=\frac{1}{2}\log\frac{(a+d)^2+(b-c)^2}{\|Z\|^2}=\frac{1}{2}\log\frac{\|Z\|^2}{(a-d)^2+(b+c)^2}=\frac{1}{4}\log\frac{(a+d)^2+(b-c)^2}{(a-d)^2+(b+c)^2}$$

(Polar coordinate representation No. 2)

Since $a^2+b^2+c^2+d^2=\left(\alpha_i^2+\beta_i^2\right)\left(\alpha_j{}^2+\beta_j{}^2\right)\left(\alpha_{ij}{}^2+\beta_{ij}{}^2\right)$,

if $Z\neq 0$, then $Z=|Z|\dfrac{\alpha_i+\beta_i i}{\sqrt{\alpha_i^2+\beta_i^2}}\dfrac{\alpha_j+\beta_j j}{\sqrt{\alpha_j{}^2+\beta_j^2}}\dfrac{\alpha_{ij}+\beta_{ij}ij}{\sqrt{\alpha_{ij}{}^2+\beta_{ij}^2}}$

There exist $\varphi_i,\varphi_j,\varphi_{ij}$ satisfying the above conditions:

$$Z=|Z|\left(\cos\varphi_i+i\sin\varphi_i\right)\left(\cos\varphi_j+j\sin\varphi_j\right)\left(\cos\varphi_{ij}+ij\sin\varphi_{ij}\right)$$

$$0\le\varphi_i<2\pi,-\frac{\pi}{2}\le\varphi_j<\frac{\pi}{2},-\frac{\pi}{4}\le\varphi_{ij}<\frac{\pi}{4}$$

The following expression is possible:

$$Z=|Z|e^{i\varphi_i+j\varphi_j}\left(\cos\varphi_{ij}+ij\sin\varphi_{ij}\right)=\sqrt{2}|Z|e^{i\varphi_i+j\varphi_j}\cos\left(\varphi_{ij}-\frac{\pi}{4}ij\right)$$

$$\cos\varphi_i=\pm\frac{1}{\sqrt{2}|Z|}\sqrt{\|Z\|^2+a^2-b^2+c^2-d^2}$$

$$\sin\varphi_i=\pm\frac{1}{\sqrt{2}|Z|}\sqrt{\|Z\|^2-a^2+b^2-c^2+d^2}$$

$$\cos\varphi_j=\frac{1}{\sqrt{2}|Z|}\sqrt{\|Z\|^2+a^2+b^2-c^2-d^2}$$

$$\sin\varphi_j=\pm\frac{1}{\sqrt{2}|Z|}\sqrt{\|Z\|^2-a^2-b^2+c^2+d^2}$$

$$\cos\varphi_{ij}=\frac{1}{\sqrt{2}|Z|}\sqrt{a^2+b^2+c^2+d^2+\|Z\|^2}$$

$$\sin\varphi_{ij}=\pm\frac{1}{\sqrt{2}|Z|}\sqrt{a^2+b^2+c^2+d^2-\|Z\|^2}$$

(Function definitions)

Functions are defined as $Z=z_R+z_i i+z_j j+z_{ij}ij\left(z_R,z_i,z_j,z_{ij}\in R\right)$, but when division is involved, the condition $Z\notin \mathbf{O}_Z$ is necessary.

(Polynomial)

For a variable $Z=z_R+z_i i+z_j j+z_{ij}ij$ and constants $A_k=a_{Rk}+a_{ik}i+a_{jk}j+a_{ijk}ij(0\le k\le n)$,

$$W=A_0+A_1Z+A_2Z^2+\cdots+A_nZ^n=P(Z)$$

(Rational)

Let $P(Z),Q(Z)$ be polynomials. A rational function is $W=\dfrac{P(Z)}{Q(Z)}$

(Exponential)

$$e^Z=1+Z+\frac{Z^2}{2!}+\frac{Z^3}{3!}+\cdots$$

(Trigonometric)

$$\sin Z=Z-\frac{Z^3}{3!}+\frac{Z^5}{5!}-\cdots,\cos Z=1-\frac{Z^2}{2!}+\frac{Z^4}{4!}-\frac{Z^6}{6!}+\cdots,\tan Z=\frac{\sin Z}{\cos Z}$$

(Hyperbolic)

$$\sinh Z=\frac{1}{2}\left(e^Z-e^{-Z}\right),\cosh Z=\frac{1}{2}\left(e^Z+e^{-Z}\right),\tanh Z=\frac{\sinh Z}{\cosh Z}$$

(Complex differentiation)

(Definition)

For $Z_0=z_{R0}+z_{i0}i+z_{j0}j+z_{ij0}ij,Z_0\notin\mathbf{O}_Z,Z=z_R+z_i i+z_j j+z_{ij}ij,Z\notin\mathbf{O}_Z$, the definition is:

$$\lim_{Z\to Z_0}\frac{F(Z)-F(Z_0)}{Z-Z_0}$$

(Properties of differentiation)

$$\left(F(Z)\pm G(Z)\right)'=F'(Z)\pm G'(Z)$$

$$\left(F(Z)\cdot G(Z)\right)'=F'(Z)\cdot G(Z)+F(Z)\cdot G'(Z)$$

$$\left(\frac{F(Z)}{G(Z)}\right)' = \frac{F'(Z)\cdot G(Z) - F(Z)\cdot G'(Z)}{(G(Z))^2}$$

$$W = f(Z), Z = G(U) \Rightarrow \frac{dW}{dU} = \frac{dW}{dZ}\frac{dZ}{dU}$$

(Cauchy-Riemann relationships)

When $Z = z_R + z_i i + z_j j + z_{ij} ij$ corresponds to $W = w_R + w_i i + w_j j + w_{ij} ij$, due to the function F, we have $W = F(Z)$. Assume that each element (w_R, w_i, w_j, w_{ij}) of point W can be expressed with functions f_R, f_i, f_j, f_{ij} of z_R, z_i, z_j, z_{ij}. That is, let

$w_R = f_R(z_R, z_i, z_j, z_{ij}), w_i = f_i(z_R, z_i, z_j, z_{ij}), w_j = f_j(z_R, z_i, z_j, z_{ij}), w_{ij} = f_{ij}(z_R, z_i, z_j, z_{ij})$. Then:

$$\begin{cases} \frac{\partial F}{\partial Z} = \frac{\partial f_R}{\partial z_R} + i\frac{\partial f_i}{\partial z_R} + j\frac{\partial f_j}{\partial z_R} + ij\frac{\partial f_{ij}}{\partial z_R} \\ \frac{\partial F}{\partial Z} = \frac{\partial f_i}{\partial z_i} - i\frac{\partial f_R}{\partial z_i} + j\frac{\partial f_{ij}}{\partial z_i} - ij\frac{\partial f_j}{\partial z_i} \\ \frac{\partial F}{\partial Z} = \frac{\partial f_j}{\partial z_j} + i\frac{\partial f_{ij}}{\partial z_j} - j\frac{\partial f_R}{\partial z_j} - ij\frac{\partial f_i}{\partial z_j} \\ \frac{\partial F}{\partial Z} = \frac{\partial f_{ij}}{\partial z_{ij}} - i\frac{\partial f_j}{\partial z_{ij}} - j\frac{\partial f_i}{\partial z_{ij}} + ij\frac{\partial f_R}{\partial z_{ij}} \end{cases} \qquad \begin{cases} \frac{\partial F}{\partial Z} = \frac{\partial f_R}{\partial z_R} - i\frac{\partial f_R}{\partial z_i} - j\frac{\partial f_R}{\partial z_j} + ij\frac{\partial f_R}{\partial z_{ij}} \\ \frac{\partial F}{\partial Z} = \frac{\partial f_i}{\partial z_i} + i\frac{\partial f_i}{\partial z_R} - j\frac{\partial f_i}{\partial z_{ij}} - ij\frac{\partial f_i}{\partial z_j} \\ \frac{\partial F}{\partial Z} = \frac{\partial f_j}{\partial z_j} - i\frac{\partial f_j}{\partial z_{ij}} + j\frac{\partial f_j}{\partial z_R} - ij\frac{\partial f_j}{\partial z_i} \\ \frac{\partial F}{\partial Z} = \frac{\partial f_{ij}}{\partial z_{ij}} + i\frac{\partial f_{ij}}{\partial z_j} + j\frac{\partial f_{ij}}{\partial z_i} + ij\frac{\partial f_{ij}}{\partial z_R} \end{cases}$$

(Jacobian)

Suppose that a small region $\mathbf{A}_Z$ in complex space $\mathbf{Z}$ is mapped to a small region $\mathbf{A}_W$ in complex space $\mathbf{W}$ by $W = F(Z)$. If functions f_R, f_i, f_j, f_{ij} are differentiable with respect to z_R, z_i, z_j, z_{ij}, then: $\lim \frac{\mathbf{A}_W}{\mathbf{A}_Z} = \frac{\partial(f_R, f_i, f_j, f_{ij})}{\partial(z_R, z_i, z_j, z_{ij})}$ holds.

When $Z = \|Z\|(\cos\theta_i + i\sin\theta_i)(\cos\theta_j + j\sin\theta_j)(\cosh\theta_{ij} + ij\sinh\theta_{ij}) = \|Z\| e^{i\theta_i + j\theta_j + ij\theta_{ij}}$, the Jacobian is

$$\frac{\partial Z}{\partial(\|Z\|, \theta_i, \theta_j, \theta_{ij})} = \|Z\|^3$$

When $Z = |Z|(\cos\varphi_i + i\sin\varphi_i)(\cos\varphi_j + j\sin\varphi_j)(\cos\varphi_{ij} + ij\sin\varphi_{ij}) = |Z| e^{i\varphi_i + j\varphi_j}(\cos\varphi_{ij} + ij\sin\varphi_{ij})$, the Jacobian is $\frac{\partial Z}{\partial(|Z|, \varphi_i, \varphi_j, \varphi_{ij})} = |Z|^3 \cos 2\varphi_{ij}$

(Sample application: Volume of a 4-dimensional sphere)

$$V = \int_{|Z|\le R} dZ$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}{}_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^R \frac{\partial Z}{\partial(r, \varphi_i, \varphi_j, \varphi_{ij})} dr d\varphi_i d\varphi_j d\varphi_{ij} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^R r^3 \cos 2\varphi_{ij} dr d\varphi_i d\varphi_j d\varphi_{ij}$$

$$= \frac{R^4}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos 2\varphi_{ij} d\varphi_i d\varphi_j d\varphi_{ij}$$

$$
\begin{aligned}
&= \frac{\pi R^4}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\varphi_{ij} \, d\varphi_j \, d\varphi_{ij} \\
&= \frac{\pi^2 R^4}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\varphi_{ij} \, d\varphi_{ij} \\
&= \frac{\pi^2 R^4}{4} \left[\sin 2\varphi_{ij} \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \frac{\pi^2 R^4}{2}
\end{aligned}
$$