A Study of Integral Notation

Keichi SUZUKI
Yamagata Prefecture,Japan
szk_kei@yahoo.co.jp

1. Introduction

In the derivative symbol $\frac{dy}{dx}$, *dx* and *dy* signify infinitesimal differences in *x* and *y*, respectively. In the definite integral notation, we have $\lim_{n\to\infty}\sum_{k=1}^{n} f\left(\xi_k\right)\left(x_i - x_{i-1}\right) = \int_a^b \left(f\left(x\right)\times dx\right) = \int_a^b f\left(x\right)dx$, and thus the dx in $\int_a^b \left(f\left(x\right)\times dx\right)$ also signifies an infinitesimal difference in x. On the other hand, in the indefinite integral notation $\int ydx$, the x in dx is called the variable of integration. This paper examines the validity of different terminology infinitesimal difference and variable of integration for the same symbol dx.

In addition, it is said that Leibniz "developed the integral symbol $\int$ by vertically elongating the first letter s of summa." The credibility of this theory is investigated.

2. Content of study

2.1 Validity of the term "variable of integration"

2.1.1 Investigation of Leibniz's achievement

In July 1676, the statement "$\frac{d\bar{x}}{d\bar{y}} = \frac{n}{y-x}$, $d\bar{x}y - xd\bar{x} = d\bar{y}n$, $\int d\bar{x}y - \int xd\bar{x} = n\int d\bar{y}$" was written as part of the answer to the problem of Methodus Tangentis Inversae (Methodus Tangentium Inversa) in Gottfried Wilhelm Leibniz(2008, PP. 598-602). Leibniz used the symbol "⊓" to mean equals, but Gerhardt rewrote this as "=". The method of turning $\frac{d\bar{x}}{d\bar{y}} = \frac{n}{y-x}$ into $d\bar{x}y - xd\bar{x} = d\bar{y}n$ involves multiplying both sides by $d\bar{y}\left(y-x\right)$, so $d\bar{x}y - xd\bar{x} = d\bar{y}n$ has the meaning that $d\bar{x}\times y - x\times d\bar{x} = d\bar{y}\times n$. With the difference equation $d\bar{x}\times y - x\times d\bar{x} = d\bar{y}\times n$, the two sides remain equal when each term is summed, i.e., $\int d\bar{x}\times y - \int x\times d\bar{x} = n\int d\bar{y}$. Since the method for finding area in this era was non-separability, Leibniz thought that "the sum of line segments becomes the area." Therefore, the meaning of $d\bar{x}\times y$ in the integral notation can be thought of as "thickness of the line segment × length of the line segment." When interpreted through Riemann sums, which are the modern definition of the integral, "$\int ydx$ is the sum of the formula $y\times dx$ of infinitesimal differences."This is the meaning of the integral notation.

On the other hand, in 1695, in C.I.Gerhardt(1863, PP. 58-59), the relationship of the derivative and integral is written as " $B_1C_1 + D_2C_2 + D_3C_3 + D_4C_4 = y$ ……The sum of the differences of y becomes y itself. (summa differentiarum inter ipeas y reddit ipsum terminum y)". Expressed in modern terms, this means that when you integrate a differentiated function, it returns the original function, i.e., $\int \frac{dy}{dx} dx = y$. Incidentally, there is also the statement that "The differential between AB_1, AB_2 is B_1B_2 or C_1D_2 … (Differentia inter duas proximas abscissas AB_1 et AB_2 est B_1B_2 seu C_1D_2 …) ". For this reason, it is thought that Leibniz changed his way of thinking about dx as the width of an infinitesimal interval. The figure below is taken from the Japanese translation of the document Gottfried Wilhelm Leibniz (1997, P. 37).

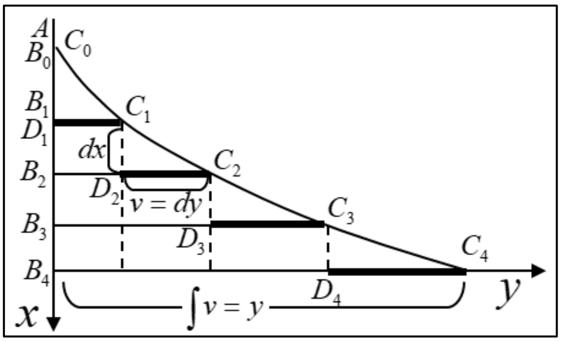


Fig. 1: Relationship between infinitesimal differences and integration

To summarize: there is no explanation in the literature about the origin of the combination of $\int$ and d in the integral notation, but there is an explanation that integration is the inverse operation of differentiation.

2.1.2 Investigation of the integrals treated in high school

There are two methods of calculation taught in high school: the inverse operation of differentiation and the method of the Riemann sum. The method of the Riemann sum for defining the definite integral was developed by Riemann in 1854. In William Dunham(2005, PP.96-115), there is the statement " $s = \delta_1 f(a + \varepsilon_1\delta_1) + \delta_2 f(a + \varepsilon_2\delta_2) + \cdots \delta_n f(a + \varepsilon_n\delta_n)$ ", which is simplified and expressed as a limit value $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$. At the time, there was the problem of whether Fourier coefficients necessarily exist, in other words, does $\int_a^b f(x)dx$ exist for any function? According to William Dunham (2005, PP.96-115), the purpose of the Riemann sum was to prove the necessary and sufficient conditions for the existence of $\int_a^b f(x)dx$. However, it is not stated to "do the calculation with the Riemann sum." In other words, the calculation method using the Riemann sum is good as calculation practice, but historically it is not a method that was required.

2.1.3 Investigation of integrals treated in university

Integrals treated in university are more advanced than high school integrals. There are double integrals, curvilinear integrals, complex integrals, surface integrals, vector function integrals, and so forth. These integrals are called Riemann integrals and are defined by the Riemann sum. However, the specific calculation methods for function integrals involve substituting variables (integration by substitution, etc.) and performing the inverse operation of differentiation.

2.1.4 Discussion of the appropriateness of the terminology "variable of integration"

In Leibniz's way of thinking, and in the integrals taught in high school and university, the calculation method emphasizes the "inverse operation of differentiation" rather than the origin of the notation. From the perspective of operation notation, when $\int_a^b f(x)\,dx$ is expressed in words, it becomes "integrate $f(x)$ with respect to x," so it is appropriate to call the x in dx the variable of integration.

2.2 Credibility of story that Leibniz developed the integral symbol $\int$ by vertically elongating the first letter "S" of summa

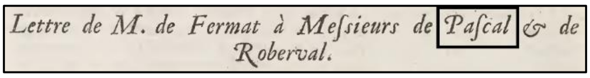
In the Japanese translation of the reference Gottfried Wilhelm Leibniz (1997, P.165), there is a translator's comment stating "This is a script that used, at the time, the first letter "S" of the Latin word summa (sum)"（これはラテン語 summa（和）の頭文字 S の当時用いられていた書体である）. The credibility of this comment will be investigated.

2.2.1 Investigation of documents from the 1700s

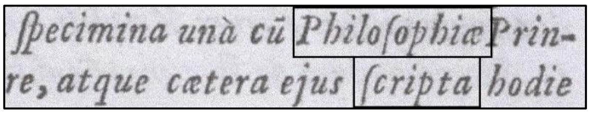
In Johannis Bernoulli(1743,P.Ⅰ), Letter (Epistola), there is the statement shown at right. The parts enclosed in rectangles are thought to correspond to "inscribe" and "philosophic."


UOS inſcribere Philoſophis

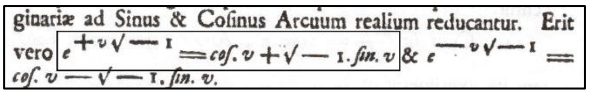
In Pierre de Fermat(1748, P.130), there is the statement shown at right. The part enclosed in the rectangle refers to the person named Pascal.


Lettre de M. de Fermat à Meſsieurs de Paſcal & de Roberval.

In René Descartes (1637, Acknowledgements (Lectori Benevolo)), there is the statement shown at right. The parts enclosed in rectangles are thought to correspond to "philosophy" and "script."


ſpecimina unà cũ Philoſophiæ Prin-
re, atque cætera ejus ſcripta hodie

In Leonhard Euler (1748, P.104), there is the statement shown at right. The part enclosed in the rectangle is Euler's formula $e^{+v\sqrt{-1}} = \cos v + \sqrt{-1}\sin v$. Each "S" in cos,sin is $\int$.


ginariæ ad Sinus & Coſinus Arcuum realium reducantur. Erit
vero $e^{+v\sqrt{-1}}$ = coſ. v + √ − 1. ſin. v & $e^{-v\sqrt{-1}}$ =
coſ. v − √ − 1. ſin. v.

2.2.2 Discussion of the origin of the integral symbol

In these references, "S" was not always stated as "all S" or "all $\int$", but as a trend, when the letter was in italics, $\int$ was used in some cases, and otherwise "S" was used. From this, it can be understood that Leibniz followed the custom of the time and made "S" into $\int$. The statement in Gottfried Wilhelm Leibniz (1997, P.165) is correct.

3. Conclusion

By closely considering the history of integration, we were able to understand the intentions hidden in Leibniz's integration formula and the points that had been misunderstood.

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