3 n-th integral function

Let $f(x)$ be an integrable function defined in the interval (x_1, x_2).
We can define

$$f^{(0)}(x) = f(x)$$

Suppose $x_0(x_1 \le x_0 \le x_2)$ is a constant,

for the variable $x(x_1 \le x \le x_2)$, we can state that $f^{(-1)}(x) = \int_{x_0}^{x} f^{(0)}(t)dt$

If $f^{(-1)}(x)$ is also integrable, we can define $f^{(-2)}(x) = \int_{x_0}^{x} f^{(-1)}(t)dt$

In the same manner, supposing $f^{(-n)}(x), (n > 0)$ is also integrable, we obtain the following definition:

$$f^{(-n-1)}(x) = \int_{x_0}^{x} f^{(-n)}(t)dt$$

Here, we call $f^{(-n)}(x)$ the n-th integral function (-n-th derived function).

The nature of $f^{(-i)}(x)$ is as follows:

if $1 \leqq i \leqq n+1$,

$$\frac{d}{dx} f^{(-i)}(x) = f^{(-i+1)}(x)$$

$$f^{(-i)}(x_0) = 0$$

If a is a constant that satisfies $a < x_1$ or $x_2 < a$,

$$\int_{x_0}^{x} \frac{f^{(0)}(t)}{t-a} dt = \left[\frac{f^{(-1)}(t)}{t-a} \right]_{x_0}^{x} + \int_{x_0}^{x} \frac{f^{(-1)}(t)}{(t-a)^2} dt$$

$$= \frac{f^{(-1)}(x)}{x-a} + \int_{x_0}^{x} \frac{f^{(-1)}(t)}{(t-a)^2} dt$$

$$\int_{x_0}^{x} \frac{f^{(-1)}(t)}{(t-a)^2} dt = \left[\frac{f^{(-2)}(t)}{(t-a)^2} \right]_{x_0}^{x} + 2\int_{x_0}^{x} \frac{f^{(-2)}(t)}{(t-a)^3} dt$$

$$= \frac{f^{(-2)}(x)}{(x-a)^2} + 2\int_{x_0}^{x} \frac{f^{(-2)}(t)}{(t-a)^3} dt$$

When i satisfies the condition $0 < i \leqq n$,

$$(i-1)!\int_{x_0}^{x}\frac{f^{(-i+1)}(t)}{(t-a)^i}dt=(i-1)!\left[\frac{f^{(-i)}(t)}{(t-a)^i}\right]_{x_0}^{x}+i!\int_{x_0}^{x}\frac{f^{(-i)}(t)}{(t-a)^{i+1}}dt$$

$$=(i-1)!\frac{f^{(-i)}(x)}{(x-a)^i}+i!\int_{x_0}^{x}\frac{f^{(-i)}(t)}{(t-a)^{i+1}}dt$$

Adding all expressions when $i=1,2,3,\ldots,n$, we get

$$\int_{x_0}^{x}\frac{f^{(0)}(t)}{t-a}dt=\sum_{i=1}^{n}(i-1)!\frac{f^{(-i)}(x)}{(x-a)^i}+n!\int_{x_0}^{x}\frac{f^{(-n)}(t)}{(t-a)^{n+1}}dt$$

When $n\to\infty$, $n!\int_{x_0}^{x}\frac{f^{(-n)}(t)}{(t-a)^{n+1}}dt\to 0$

Hence, $\int_{x_0}^{x}\frac{f^{(0)}(t)}{t-a}dt=\sum_{n=1}^{\infty}(n-1)!\frac{f^{(-n)}(x)}{(x-a)^n}$

Since the variable is a real number, we obtain the expression $\frac{d}{dx}\int_{x_0}^{x}\frac{f^{(0)}(t)}{t-a}dt=\frac{f(x)}{x-a}$, which is more complicated than the integral expression obtained when using a complex number.

Example calculation

If $x \geqq \frac{1}{2}$, $f(x)=1, a=0, x_0=1$, we get

$$\int_{x_0}^{x} \frac{f^{(0)}(t)}{t-a}dt = \int_1^x \frac{1}{t}dt = \log x$$

$$f^{(-1)}(x) = \int_1^x 1dt = x-1$$

$$f^{(-2)}(x) = \int_1^x (t-1)dt = \frac{(x-1)^2}{2}$$

$$f^{(-n)}(x) = \frac{(x-1)^n}{n!}$$

$$\sum_{i=1}^{n}(i-1)!\frac{f^{(-i)}(x)}{(x-a)^i} = \sum_{i=1}^{i}\frac{(x-1)^i}{ix^i}$$

$$n!\int_{x_0}^{x}\frac{f^{(-n)}(t)}{(t-a)^{n+1}}dt = n!\int_1^x \frac{1}{t^{n+1}}\frac{(t-1)^n}{n!}dt$$

$$= \int_1^x \frac{(t-1)^n}{t^{n+1}}dt$$

$$= \int_1^x \frac{(t-1)^n}{t^{n+1}}dt$$

$$\int_1^x \frac{(t-1)^n}{t^{n+1}}dt = \int_1^x \left(1-\frac{1}{t}\right)^n \frac{1}{t}dt$$

When $1 \leqq x$, the condition $\varepsilon > 0$ is satisfied, which is sufficiently small for the condition $n > N$, and we obtain

$$\left(1-\frac{1}{t}\right)^n < \left(1-\frac{1}{x}\right)^n < \varepsilon$$

$$\therefore \int_1^x \frac{(t-1)^n}{t^{n+1}}dt < \int_1^x \frac{\varepsilon}{t}dx = \varepsilon \log x$$

Therefore, when $n \to \infty$, $\int_1^x \frac{(t-1)^n}{t^{n+1}}dt \to 0$

$$\therefore \log x = \sum_{n=1}^{\infty}\frac{(x-1)^n}{nx^n}$$

When $\frac{1}{2} < x < 1$, $\frac{1-t}{t} < \frac{1-x}{x} < 1$. Hence, using the same proof as above,

$$n \to \infty,\ \int_1^x \frac{(t-1)^n}{t^{n+1}} dt \to 0$$

$$\therefore \log x = \sum_{n=1}^{\infty} \frac{(x-1)^n}{nx^n}$$

When $x = \frac{1}{2}$, the relation

$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ holds true. Therefore,

when $x = \frac{1}{2}$, $\log x = \sum_{n=1}^{\infty} \frac{(x-1)^n}{nx^n}$ also holds true.