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A Study of Definition of the Definite Integral

- Proposal to add a logical interpretation to definition of the definite integral to make it easier to understand -

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Abstract: In high school mathematics classes, students learn the basic properties, calculation methods, and applications of the integral. However, they don't learn the history of the integral symbol, so there are cases where they can't adequately understand, or misunderstand, the meaning of the definite integral symbol. Here, an approach to definition of the definite integral is proposed that is based on Leibniz's approach to the integral. Using this approach as a basis makes it easier to understand the meaning of the definite integral symbol, and enables understanding of the meaning of the various integration symbols learned in university.

Search terms: definition of the definite integral, meaning of symbols, Leibniz, integral

1. Objective of the study

According to Tsuneharu OKABE (2021), the definition of the definite integral using piecewise quadrature is:

$$\lim_{n\to\infty}\sum_{k=1}^{n} f\left(x_k\right)\Delta x=\int_a^b f\left(x\right)dx$$

$$\text{Where } \Delta x=\frac{b-a}{n}, x_k=a+k\Delta x$$

Incidentally, if we look closely at the definition of the definite integral, it is evident that there is a gap between the representation methods on the left and right side. The "=" sign can be interpreted as indicating definition, but if it is interpreted in the sense of "=" in an equation, then the rule corresponding to the operation rule where "left side = right side" cannot be explained. Thus, to find a rule corresponding to the operation rule between the two sides, the author decided to survey the work of Leibniz, and investigate the origin of the integration symbol. In this study, the work of Leibniz is explained, and at the same time, an approach to definition of the definite integral suggested by that is proposed.

Furthermore, students learn various integrals in university. Among those, Yasuzo FUKUDA (1976) presents the following double integral when D is bounded and $f\left(x,y\right)$ is continuous:

$$\iint_D f\left(x,y\right)dxdy=\int_c^d\left\{\int_a^b f\left(x,y\right)\right\}dy$$

$$=\int_c^d dy\int_a^b f\left(x,y\right)dx$$

Here, if we set $\int_a^b f\left(x,y\right)dx=F\left(y\right)$, then the equation

$\int_c^d F\left(y\right)dy=\int_c^d dyF\left(y\right)$ holds. In this equation, the commutative law holds between $F\left(y\right)$ and dy. Therefore, grounds for why the commutative law holds are also proposed.

2. Content of the study (survey of the work of Leibniz)

2.1 Overview of Leibniz's work

Koshiro NAKAMURA (1999, pp. 209–242) provides the following overview of the work of Leibniz:

1666

Published dissertation "Dissertatio de arte combinatoria" and "Historia et Origo." Created the symbols d and $\int$. However, these are symbols for difference and sum, and don't yet have the meaning of differentiation and integration.

1673

Discovered transmutation theorem based on the characteristic triangle. Showed the existence of a tangent line of a function, and an area function definable from the tangent lines. After this, he began to use d and $\int$ as symbols for finding a tangent line or area. However, usage of the symbols was trial and error.

1684

Published paper on differential calculus. Title: "A new method for maxima and minima, and for tangents, that is not hindered by fractional or irrational quantities..."

1686

Published paper on integral calculus. Title: "On a hidden geometry and analysis of indivisibles and infinites"

2.2 Shift in usage of d and $\int$, and discussion

In his dissertation, Leibniz treated the symbol d as difference and the symbol $\int$ as sum. After discovery of the characteristic triangle and the transmutation theorem, Leibniz worked on the problems of tangents and quadrature, and as a technique for that, he worked to establish a notation and calculation method. He began using d as a symbol for an infinitesimal difference (differential), and $\int$ as a symbol for finding area. There was trial and error in how these symbols were used.

2.2.1 Shift in usage of d and $\int$

The following is presented from Gottfried Wilhelm Leibniz (1997).

October 1675 ① (p. 150)

For the problem of quadrature, he uses the symbol "omn." devised by Cavalieri.

(Ex.) $\text{omn.}\overline{yx}\ \text{ad}\ x \sqcap \frac{b^2c}{2} - \text{omn.}\overline{\frac{x^2}{2}\ \text{ad}\ y}$

"omn." is an abbreviation for the Latin "omnes" and means "all lines" in the method of indivisibles. "$\overline{xy}$" means "(xy)". "$ad\ x$" means "related to x". "$\sqcap$" means "=".

October 1675 ② (P.165)

Leibniz writes: "it should be convenient to write $\int$ instead of omn.". The symbol $\int$ was introduced for the first time in the history of mathematics.

(Ex.) $\int x \sqcap \frac{x^2}{2}$

July 1676 (pp. 212–213)

In the problem of the inverse method of tangents of Descartes, the differential symbol was placed in the denominator for the first time: $\frac{t}{y} \sqcap \frac{d\bar{x}}{d\bar{y}}$. This equation was further transformed as follows, and applied to problem of quadrature:

$$\frac{d\bar{x}}{d\bar{y}} \sqcap \frac{n}{y-x}$$

$$d\bar{x}y - xd\bar{x} \sqcap d\bar{y}n$$

$$\int d\bar{x}y - \int \overline{xd\bar{x}} \sqcap n\int d\bar{y}$$

November 1676 (pp. 238–241)

General rules are derived for the derivative and integral of simple exponents.

$$\frac{dx^e}{dx} \sqcap ex^{e-1}, \int \overline{x^e d\bar{x}} \sqcap \frac{x^{e+1}}{e+1}$$

This is also applied to complicated irrational expressions and fractional expressions like the following.

$$d\sqrt{\overline{a++bz+cz^2}} \sqcap -\frac{b+2cz}{2\sqrt{a+bz+cz^2}}$$

October 1684, Paper on differential calculus (pp. 296–297)

$dx, dy, \cdots$ are regarded as finite line segments tangent to the curves $X, Y, \cdots$, and the operation rules between finite line segments are given.

- If a is constant $da = 0$
- $d\overline{ax} = adx$
- $d\overline{z-y+w+x} = dz - dy + dw + dx$
- $d\overline{xv} = xdv + vdx$
- $d\frac{v}{y} = \frac{\pm vdy \mp ydv}{yy}$

In this reference, the phrase "is equal to" is used instead of the equals sign "=". The "=" sign appears from p. 299 of the same reference. The term "finite line segment" is quoted from Koshiro Nakamura (1999, p. 238).

July 1686, Paper on integral calculus (pp. 326, 327)

Here, Leibniz writes, regarding the integral of a general function: "If $pdy = xdx$, and this differential equation is converted to a summation equation, it becomes $\int pdy = \int xdx$."

2.2.2 Discussion of the shift in usage of d and $\int$

Discussion of July 1676

The method of changing $\frac{d\bar{x}}{d\bar{y}} \sqcap \frac{n}{y-x}$ to $d\bar{x}y - xd\bar{x} \sqcap d\bar{y}n$ is to multiply both sides of $\frac{d\bar{x}}{d\bar{y}} \sqcap \frac{n}{y-x}$ by $d\bar{y}(y-x)$. Therefore, it can be interpreted that $d\bar{x} \times y - x \times d\bar{x} \sqcap d\bar{y} \times n$.

The equality of the difference expression $\left(d\bar{x} \times y - x \times d\bar{x}\right)$ with the difference expression $\left(d\bar{y} \times n\right)$ is indicated as $\int d\bar{x}y - \int \overline{xd\bar{x}} \sqcap n \int d\bar{y}$.

In $\int d\bar{x}y$ for example, It is appropriate to interpret the summation symbol $\int$ as not acting only on $d\bar{x}$ or only on y, but as acting on $d\bar{x} \times y$ .This can be interpreted as

$\int\left(d\bar{x} \times y\right) - \int\left(x \times d\bar{x}\right) \sqcap n \times \int d\bar{y}$.

July 1686, Discussion of the paper on integral calculus

Both sides of $\frac{dx}{dy} = \frac{p}{x}$ are multiplied by xdy, so it can be interpreted as $p \times dy = x \times dx$, and further as $\int(p \times dy) = \int(x \times dx)$.

Discussion of the fact that "•" was omitted

After the discovery of the characteristic triangle and the transmutation theorem, Leibniz struggled to realize a symbolism with d and $\int$, but in Gottfried Wilhelm Leibniz (1997), "·" is not written inside the integral symbol. It is likely that "·" was omitted. This raises the question of why Leibniz omitted "·" Incidentally, Descartes' (1596–1650) *Discourse on the Method* is presented in Koshiro Nakamura (1999, pp. 40–56). There it is written by Koshiro Nakamura (1999, P. 47) that "the four rules of arithmetic for two line segments are expressed as $a+b$, $a-b$, ab, $\frac{a}{b}$", and "·" and "$\times$" are omitted for multiplication. That is, it is likely that Leibniz adopted the notational methods of Descartes.

3. Research results

For these reasons, it is proposed to revise the definition of the definite integral as follows.

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b \left(f(x) \times dx\right)$$
$$= \int_a^b f(x)dx$$

The meaning of $\int_a^b \left(f(x) \times dx\right)$ is to find the area by multiplying the infinitesimal difference dx by the value $f(x)$ at x on an infinitesimal interval, and this is enclosed in parentheses. Also, the source of the integral is the meaning of a sum (Latin "summa"), and thus it means "find the sum from a to b". Subsequently, omitting $\times$ and $(\,)$ yields the conventional expression.

4. Properties and greater strictness of "$\times$"

Consider the properties of "$\times$" in

$\int_a^b \left(f(x) \times dx\right)$.

Integrating by substitution, this can be written as follows:

$$\int_a^b \left(f(x) \times dx\right) = \int_\alpha^\beta \left(f(g(t)) \times \frac{dx}{dt} \times dt\right)$$
$$= \int_\alpha^\beta f(g(t))\frac{dx}{dt}dt$$

Therefore, "$\times$" has the same properties as multiplication. Due to the commutative law of multiplication, the equation

$\int_a^b \left(f(x) \times dx\right) = \int_a^b \left(dx \times f(x)\right)$ holds.

Now, in the previous section, dx was explained as an infinitesimal difference, but the question arises: "how great is the difference of an infinitesimal difference?" If, in contrast, a decimal description is used, then the number of zeroes after the decimal point is infinite, and it's impossible to specify a precise value of the infinitesimal difference. Therefore, an infinitesimal difference is an abstract concept. Operations are done on an abstract infinitesimal difference, so " × " is multiplication of abstract concepts.

5. Learning effect due to understanding the definition

Now let us explain the learning effect due to incorporating the definition of the definite integral explained in this paper into the learning process of high school students.

When high school students learn the definition of the definite integral based on piecewise quadrature, incorporating the material explained in this paper is anticipated to enable understanding, without any stress, of the sophisticated integrals that will be learned if the student proceeds on to university in a scientific or technical field. They should be able to understand the meaning of symbols for double integrals, surface integrals, line integrals, complex integrals, integrals on vectors, and (in mathematics) Lebesgue integrals. Also, with previous instructional methods, there is no need to adequately explain the definition of the definite integral, and thus there is a possibility that some high school students do not correctly understand the definition part. For this sort of high school student, it is expected that using the method explained in this study will enable them to correctly understand the definition of the definite integral.

Quote and reference bibliography

Gottfried Wilhelm Leibniz;
Kokichi HARA, Chikara SASAKI, Nobuo MIURA, Kaoru BABA, Ken SAITO, Masahito ANDO, Takashi KURATA (translators),
Leibniz Collected Works 2, Mathematical Theory and Mathematics, Kousakusha, 1997 (in Japanese)

Gottfried Wilhelm Leibniz;
Kokichi HARA, Masahiko YOKOYAMA, Nobuo MIURA, Kaoru BABA, Takashi KURATA, Hideo NAGASHIMA, Keisho NISHI (translators),
Leibniz Collected Works 3, Mathematics and Natural Science, Kousakusha, pp. 310–315, 1999 (in Japanese)

Tsuneharu OKABE, Kazushi AHARA, Kazuhiro ICHIHARA, Shunsuki IHARA, Masanori OHYA, Mitsuyuki OCHIAI, Nobuyuki TOSE, Yatsuka NAKAMURA, Nobuyoshi MOTOHASHI, Jun MORITA, Katsumi YAGI, Masaaki YOSHIDA, Haruki KATO, Mayumi TACHI, Mamoru NISHIMAKI, Keiko YAMAMOTO, Yoshihiro KICHIJI, Suken Shuppan Editorial Staff
High School Mathematics III, Suken Shuppan, p. 227, 2021 (in Japanese)

Koshiro NAKAMURA, *History of Modern Mathematics*, Nippon Hyoron Sha Co., Ltd., 1999 (in Japanese)

Yasuzo FUKUDA, Nanao SUZUKI, Yoshinori YASUOKA, Chiyoko KUROSAKI
Differential and Integral Calculus Exercises II, Kyoritsu Shuppan, p. 71, 1976 (in Japanese)